

# $q$ -Virasoro/ $W$ Algebra at Root of Unity and Parafermions

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## Abstract

We demonstrate that the parafermions appear in the  $r$ -th root of unity limit of  $q$ -Virasoro/ $W_n$  algebra. The proper value of the central charge of the coset model  $\frac{\widehat{\mathfrak{sl}(n)}_r \oplus \widehat{\mathfrak{sl}(n)}_{m-n}}{\widehat{\mathfrak{sl}(n)}_{m-n+r}}$  is given from the parafermion construction of the block in the limit.

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# 1 Introduction

Ever since the AGT relation [1, 2, 3] (the correspondence between the correlators of 2d QFT and the 4d instanton sum) was introduced, the both sides of the correspondence have been intensively studied by a number of people. For example, in the 2d side, the  $\beta$ -deformed matrix model is used in order to control the integral representation of the conformal block [4, 5, 6, 7, 8, 9, 10]. There are also some proposals for proving the 2d-4d connection [11, 12, 13, 14, 15]. Moreover similar correspondence has been found and examined [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Among these, we pay our attention, in this paper, to the correspondence between the coset model,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_p}{\widehat{\mathfrak{sl}}(n)_{r+p}}, \quad (1.1)$$

and the  $\mathcal{N} = 2$   $SU(n)$  gauge theory on  $\mathbf{R}^4/\mathbf{Z}_r$  [20, 23]. Here  $\widehat{\mathfrak{sl}}(n)_k$  stands for the affine Lie algebra in the representation of level  $k$  and  $r$  and  $p$  will be specified in this paper.

On the 2d CFT side, a quantum deformation ( $q$ -deformation) of the Virasoro algebra [27] and the  $W_n$  algebra [28, 29] is known, while the 4d gauge theories can be lifted to five-dimensional theories with the fifth direction compactified on a circle. There exists a natural generalization to the connection between the 2d theory based on the  $q$ -deformed Virasoro/ $W$  algebra and the five-dimensional  $\mathcal{N} = 2$  gauge theory [30]. For recent developments, see, for example, [31, 32, 33, 34, 35, 36, 37]. In the previous paper [32], we proposed a limiting procedure to get the Virasoro/ $W$  block in the 2d side from that in the  $q$ -deformed version. On the other hand, we saw that the instanton partition function on  $\mathbf{R}^4/\mathbf{Z}_r$  are generated from that on  $\mathbf{R}^5$  at the same limit. This result means if we assume the 2d-5d connection, it is automatically assured that the Virasoro/ $W$  blocks generated by using the limiting procedure agree with the instanton partition function on  $\mathbf{R}^4/\mathbf{Z}_r$ . Our limiting procedure corresponds to a root of unity limit in  $q$ . A root of unity limit of the  $q$ -Virasoro algebra was also considered in [38]. Our limit is slightly different from this and is similar to the one used in order to construct the eigenfunctions of the spin Calogero-Sutherland model from Macdonald polynomials in [39, 40].

In the present paper we will elaborate our limiting procedure and show that the  $\mathbf{Z}_r$ -parafermionic CFT which has the symmetry described by (1.1) appears in the 2d side. We clarify also the relation between the free parameter  $p$  and the omega background parameters in the 4d side.

The paper is organized as follows: In the next section, we review the limiting procedure for  $q$ -Virasoro algebra [32]. In section 3, we consider the  $q$ -deformed screening current and charge and show that the  $\mathbf{Z}_r$ -parafermion currents are derived in a natural way. In section 4, we consider the generalization to  $q$ - $W_n$  algebra.

## 2 Root of Unity Limit of $q$ -Virasoro Algebra

In this section, we review the root of unity limit [32] of the  $q$ -deformed Virasoro algebra [27] which has two parameters  $q$  and  $t = q^\beta$ . The defining relation is

$$f(z'/z)\mathcal{T}(z)\mathcal{T}(z') - f(z/z')\mathcal{T}(z')\mathcal{T}(z) = \frac{(1-q)(1-t^{-1})}{(1-p)} \left[ \delta(pz/z') - \delta(p^{-1}z/z') \right], \quad (2.1)$$

where  $p = q/t$  and

$$f(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} z^n \right). \quad (2.2)$$

The multiplicative delta function is defined by

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n. \quad (2.3)$$

Using the  $q$ -deformed Heisenberg algebra  $\mathcal{H}_{q,t}$ :

$$\begin{aligned} [\alpha_n, \alpha_m] &= -\frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} \delta_{n+m,0}, \quad (n \neq 0), \\ [\alpha_n, Q] &= \delta_{n,0}, \end{aligned} \quad (2.4)$$

the  $q$ -Virasoro operator  $\mathcal{T}(z)$  can be realized as

$$\mathcal{T}(z) = : \exp \left( \sum_{n \neq 0} \alpha_n z^{-n} \right) : p^{1/2} q^{\sqrt{\beta} \alpha_0} + : \exp \left( - \sum_{n \neq 0} \alpha_n (pz)^{-n} \right) : p^{-1/2} q^{-\sqrt{\beta} \alpha_0}, \quad (2.5)$$

The  $q$ -deformed chiral bosons are defined in terms of the  $q$ -deformed Heisenberg algebra as

$$\tilde{\varphi}^{(\pm)}(z) = \tilde{\varphi}_0^{(\pm)}(z) + \tilde{\varphi}_R^{(\pm)}(z), \quad (2.6)$$

where

$$\begin{aligned} \tilde{\varphi}_0^{(\pm)}(z) &= \beta^{\pm 1/2} Q + \frac{2}{r} \beta^{\pm 1/2} \alpha_0 \log z^r + \sum_{n \neq 0} \frac{(1+p^{-nr})}{(1-\xi_{\pm}^{nr})} \alpha_{nr} z^{-nr}, \\ \tilde{\varphi}_R^{(\pm)}(z) &= \sum_{\ell=1}^{r-1} \sum_{n \in \mathbb{Z}} \frac{(1+p^{-nr-\ell})}{1-\xi_{\pm}^{nr+\ell}} \alpha_{nr+\ell} z^{-nr-\ell}. \end{aligned} \quad (2.7)$$

Here  $\xi_+ = q$ ,  $\xi_- = t$ .

Let us consider the simultaneous  $r$ -th root of unity limit in  $q$  and  $t$  which is given by

$$q = \omega e^{-\frac{1}{\sqrt{\beta}} h}, \quad t = \omega e^{-\sqrt{\beta} h}, \quad p = e^{Q_E h}, \quad h \rightarrow 0, \quad (2.8)$$

where  $\omega = e^{\frac{2\pi i}{r}}$  and  $Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$ . Since  $t = q^\beta$ , this limit is possible if the parameter  $\beta$  takes the rational number such as

$$\beta = \frac{rm_- + 1}{rm_+ + 1}, \quad (2.9)$$

where  $m_{\pm}$  are non-negative integers. In the limit, we have two types of bosons  $\phi(w)$  and  $\varphi(w)$  [32] respectively given by

$$\begin{aligned} \lim_{h \rightarrow 0} \tilde{\varphi}_0^{(\pm)}(z) &= \sqrt{\frac{2}{r}} \beta^{\pm 1/2} \phi(w), \\ \lim_{h \rightarrow 0} \tilde{\varphi}_R^{(\pm)}(z) &= \sqrt{\frac{2}{r}} \varphi(w), \end{aligned} \quad (2.10)$$

where  $w = z^r$  and

$$\phi(w) = Q_0 + a_0 \log w - \sum_{n \neq 0} \frac{a_n}{n} w^{-n}, \quad (2.11)$$

$$\varphi(w) = \sum_{\ell=1}^{r-1} \varphi^{(\ell)}(w), \quad \varphi^{(\ell)}(w) = \sum_{n \in \mathbb{Z}} \frac{\tilde{a}_{n+\ell/r}}{n + \ell/r} w^{-n-\ell/r}. \quad (2.12)$$

The commutation relations are

$$\begin{aligned} [a_m, a_n] &= m \delta_{m+n,0}, & [a_n, Q_0] &= \delta_{n,0}, \\ [\tilde{a}_{n+\ell/r}, \tilde{a}_{-m-\ell'/r}] &= (n + \ell/r) \delta_{m,m'} \delta_{\ell,\ell'}. \end{aligned} \quad (2.13)$$

The boson  $\phi(w)$  and the twisted boson  $\varphi(w)$  play an important role for the appearance of the  $\mathbf{Z}_r$ -parafermions.

### 3 $\mathbf{Z}_r$ -parafermionic CFT

The  $q$ -deformed screening current and the charge are defined respectively by

$$S^{(\pm)}(z) =: e^{\tilde{\varphi}^{(\pm)}(z)} :, \quad Q_{[a,b]}^{(\pm)} = \int_a^b d\xi_{\pm} z S^{(\pm)}(z), \quad (3.1)$$

where the Jackson integral is defined by

$$\int_0^a d\xi_{\pm} z f(z) = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k. \quad (3.2)$$

Multiplying the regularization factor, we obtain the screening charge in the root of unity limit, up to normalization,

$$Q_{[a^r, b^r]}^{(\pm)} \equiv \lim_{h \rightarrow 0} \frac{(1-q^r)}{(1-q)} Q_{[a,b]}^{(\pm)} = \int_{a^r}^{b^r} dw \psi_1(w) : e^{\sqrt{B}\phi(w)}, \quad (3.3)$$

where we have defined [41]

$$\psi_1(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{k=0}^{r-1} \omega^k : \exp \left\{ \sqrt{\frac{2}{r}} \phi^{(k)}(w) \right\} :. \quad (3.4)$$

Here  $A_r$  is the normalization factor and we have introduced

$$\phi^{(k)}(w) \equiv \varphi(e^{2\pi i k} w). \quad (3.5)$$

The correlation function is given by

$$\langle \phi^{(k)}(w) \phi^{(k')}(w') \rangle = \log \frac{(1 - \omega^{k'-k} (w'/w)^{1/r})^r}{1 - w'/w} = \log \frac{(1 - w'/w)^{r-1}}{\prod_{j=1}^{r-1} (1 - \omega^{k'-k+j} (w'/w)^{1/r})^r}. \quad (3.6)$$

Note that

$$\phi^{(k+1)}(w) = \phi^{(k)}(e^{2\pi i}w), \quad \phi^{(r+k)}(w) = \phi^{(k)}(w), \quad \sum_{k=0}^{r-1} \phi^{(k)}(w) = 0. \quad (3.7)$$

For example, we consider the  $r = 2$  case. In the limit, we obtain

$$\lim_{q \rightarrow -1} S(z) =: e^{\sqrt{\beta}\phi(w)} e^{\varphi(w)} :, \quad (3.8)$$

and after the appropriate normalization, we obtain the following screening charge for the superconformal block [42, 43]:

$$Q_{[a^2, b^2]} = \int_{a^2}^{b^2} dw \psi(w) : e^{\sqrt{\beta}\phi(w)} :, \quad (3.9)$$

where

$$\psi(w) \equiv \frac{i}{2\sqrt{2w}} \left( : e^{\varphi(w)} : - : e^{-\varphi(w)} : \right), \quad \langle \psi(w_1) \psi(w_2) \rangle = \frac{1}{w_1 - w_2}, \quad (3.10)$$

is the NS fermion.

From now on we will show that the  $\mathbf{Z}_r$ -parafermions appear in the general  $r$ -th root of unity limit. In particular,  $\psi_1(w)$  will be shown to work as the first parafermion current.

The  $\mathbf{Z}_r$ -parafermion algebra consists of  $(r-1)$  currents  $\psi_\ell(w)$  ( $\ell = 1, \dots, r-1$ ) satisfying the following defining relations [44]:

$$\psi_\ell(w) \psi_{\ell'}(w') = \frac{c_{\ell, \ell'}}{(w - w')^{2\ell\ell'/r}} \{ \psi_{\ell+\ell'}(w') + \mathcal{O}(w - w') \}, \quad \ell + \ell' < r, \quad (3.11)$$

$$\psi_\ell(w) \psi_{\ell'}^\dagger(w') = c_{\ell, r-\ell'} (w - w')^{-2\ell(r-\ell')/r} \{ \psi_{\ell-\ell'}(w') + \mathcal{O}(w - w') \}, \quad \ell' < \ell \quad (3.12)$$

$$\psi_\ell(w) \psi_\ell^\dagger(w') = (w - w')^{-2\Delta_\ell} \left\{ 1 + \frac{2\Delta_\ell}{c_p} (w - w')^2 T_{\text{PF}}(w) + \mathcal{O}((w - w')^3) \right\}, \quad (3.13)$$

where  $\psi_\ell^\dagger(w) = \psi_{r-\ell}(w)$  and

$$\Delta_\ell = \frac{\ell(r-\ell)}{r}, \quad c_p = \frac{2(r-1)}{r+2}, \quad (3.14)$$

are the conformal dimension of  $\psi_\ell(w)$  and the central charge of the parafermionic stress tensor  $T_{\text{PF}}$ . The explicit form of  $T_{\text{PF}}(w)$  is given in [45]. The coefficients  $c_{\ell\ell'}$  are given by

$$c_{\ell\ell'} = \sqrt{\frac{(\ell + \ell')!(r - \ell)!(r - \ell')!}{\ell!\ell'!(r - \ell - \ell')!r!}}. \quad (3.15)$$

The OPE of (3.4) is

$$\psi_1(w) \psi_1(w') \equiv \frac{c_{1,1}}{(w - w')^{2/r}} \{ \psi_2(w) + \mathcal{O}(w - w') \}. \quad (3.16)$$

Here we have defined the second parafermion,

$$\psi_2(w) = \frac{A_r^2}{c_{1,1} w^{2(r-2)/r}} \sum_{k, k'=0}^{r-1} \omega^{k+k'} (1 - \omega^{k'-k})^2 : e^{\sqrt{\frac{2}{r}}(\phi^{(k)}(w) + \phi^{(k')}(w))} : \quad (3.17)$$

Similarly, the  $(\ell + 1)$ -th parafermion is obtained from  $\ell$ -th parafermion by

$$\psi_{\ell+1}(w) \equiv \lim_{w' \rightarrow w} \frac{(w - w')^{2\ell/r}}{c_{1,\ell}} \psi_1(w') \psi_\ell(w). \quad (3.18)$$

In particular,

$$\psi_1^\dagger(w) \equiv \psi_{r-1}(w) = \frac{B_r}{w^{(r-1)/r}} \sum_{\ell=0}^{r-1} \omega^\ell \exp \left\{ -\sqrt{\frac{2}{r}} \phi^{(\ell)}(w) \right\}, \quad (3.19)$$

where  $B_r$  is a constant which can be determined by the relation

$$\langle \psi_1(w) \psi_1^\dagger(w') \rangle = \frac{1}{(w - w')^{2(r-1)/r}}. \quad (3.20)$$

After all, we have the chiral boson  $\phi(w)$  coupled to  $Q_E$  and the  $\mathbf{Z}_r$ -parafermion  $\psi_\ell(w)$ . Therefore, the stress tensor of the whole system is

$$T(w) = T_B(w) + T_{\text{PF}}(w), \quad (3.21)$$

where  $T_B(w)$  stands for the usual stress tensor for the chiral boson field. The central charge is

$$c^{(r)} = 1 - \frac{6Q_E^2}{r} + \frac{2(r-1)}{r+2} = \frac{3r}{r+2} - \frac{6Q_E^2}{r}. \quad (3.22)$$

Because  $\beta$  is restricted to the rational number (2.9), (3.22) is written as

$$c^{(r,m,s)} = \frac{3r}{r+2} - \frac{6rs^2}{m(m+rs)}. \quad (3.23)$$

where we have set  $m = rm_+ + 1$  and  $s = m_- - m_+$ . Especially, when  $s = 1$ ,

$$c^{(r,m,1)} = \frac{3r}{r+2} - \frac{6r}{m(m+r)}, \quad (3.24)$$

is the central charge of the unitary series of the  $\mathbf{Z}_r$ -parafermionic CFT [46].

The form of the screening charge in the case of general  $r$  is the same as that of eq. (3.9).

## 4 Root of Unity Limit of $q$ - $W_n$ Algebra

In this section, we consider the generalization to the  $q$ - $W_n$  algebra [29]. We denote by  $\mathfrak{h}$  the Cartan subalgebra of  $\mathfrak{sl}(n)$  Lie algebra. The  $q$ - $W_n$  algebra is expressed in terms of the following  $\mathfrak{h}$ -valued  $q$ -deformed boson,

$$\langle e_a, \tilde{\varphi}^{(\pm)}(z) \rangle \equiv \tilde{\varphi}_a^{(\pm)}(z) = \tilde{\varphi}_{0,a}^{(\pm)}(z) + \tilde{\varphi}_{R,a}^{(\pm)}(z), \quad (4.1)$$

where

$$\tilde{\varphi}_{0,a}^{(\pm)}(z) = \beta^{\pm\frac{1}{2}} Q_a + \beta^{\pm\frac{1}{2}} \alpha_{0,a} \log z + \sum_{n \neq 0} \frac{1}{\xi_{\pm}^{nr/2} - \xi_{\pm}^{-nr/2}} \alpha_{nr,a} z^{-nr}, \quad (4.2)$$

$$\tilde{\varphi}_{R,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1} \tilde{\varphi}_{\ell,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1} \sum_{n \in \mathbf{Z}} \frac{1}{\xi_{\pm}^{(nr+\ell)/2} - \xi_{\pm}^{-(nr+\ell)/2}} \alpha_{nr+\ell,a} z^{-(nr+\ell)}, \quad (4.3)$$

and  $e_a$  ( $a = 1, \dots, n-1$ ) are the simple roots and  $\langle, \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \rightarrow \mathbf{C}$  is the canonical pairing. The commutation relations are given by

$$\begin{aligned} [Q_a, \alpha_{0,b}] &= C_{ab}, \\ [\alpha_{n,a}, \alpha_{m,b}] &= \frac{1}{n}(q^{n/2} - q^{-n/2})(t^{n/2} - t^{-n/2})C_{ab}(p)\delta_{n+m,0}, \\ [Q_a, Q_b] &= 0, \quad [\alpha_{0,a}, \alpha_{0,b}] = 0, \end{aligned} \quad (4.4)$$

where  $C_{ab}$  is the Cartan matrix of  $A$  type and

$$C_{ab}(p) = [2]_p \delta_{a,b} - p^{1/2} \delta_{a,b-1} - p^{-1/2} \delta_{a-1,b}. \quad (4.5)$$

The  $q$ -number is defined by

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \quad (4.6)$$

Similar to the  $q$ -Virasoro case, we consider the limit,

$$q = \omega^k e^{-\frac{h}{\sqrt{\beta}}}, \quad t = \omega^k e^{-\sqrt{\beta}h}, \quad p = q/t = e^{Q_E h}, \quad \omega = e^{\frac{2\pi i}{r}}, \quad h \rightarrow +0, \quad (4.7)$$

where  $\omega = e^{\frac{2\pi i}{r}}$  and  $k$  is a natural number mutually prime to  $r$ . The condition to be able to take this limit is that  $\beta$  is a rational number,

$$\beta = \frac{rm_- + k}{rm_+ + k}, \quad (4.8)$$

where  $m_{\pm}$  are non-negative integers. Taking this limit,

$$\lim_{h \rightarrow 0} \tilde{\varphi}_0^a(z) = \frac{1}{\sqrt{r}} \beta^{1/2} \phi^a(w), \quad (4.9)$$

$$\lim_{h \rightarrow 0} \tilde{\varphi}_R^a(z) = \frac{1}{\sqrt{r}} \varphi^a(w), \quad (4.10)$$

we obtain

$$\phi^a(w) = Q_0^a + a_0^a \log w - \sum_{n \neq 0} \frac{1}{n} a_n^a w^{-n}, \quad (4.11)$$

$$\varphi^a(w) = \sum_{\ell=1}^{r-1} \varphi_{\ell}(w), \quad \varphi_{\ell}(w) = \sum_{\ell=1}^{r-1} \sum_{n \in \mathbf{Z}} \frac{1}{n + \ell/r} \tilde{a}_{n+\ell/r}^a w^{-(n+\ell/r)}, \quad (4.12)$$

Here we have normalized as

$$Q^a = \frac{1}{\sqrt{r}} Q_0^a, \quad \alpha_0^a = \sqrt{r} a_0^a, \quad (4.13)$$

$$\alpha_{nr}^a = -(-1)^{nk} \sqrt{r} h a_n^a, \quad (4.14)$$

$$\alpha_{nr+\ell}^a = \frac{e^{i\pi k(nr+\ell)/2} - e^{-i\pi k(nr+\ell)/2}}{\sqrt{r}(n + \ell/r)} \tilde{a}_{n+\ell/r}^a. \quad (4.15)$$

The commutation relations are

$$[Q^a, \alpha_0^b] = C_{ab}, \quad [Q^a, Q^b] = 0, \quad [\alpha_0^a, \alpha_0^b] = 0, \quad (4.16)$$

$$[a_n^a, a_m^b] = nC_{ab}\delta_{n+m,0}, \quad (4.17)$$

$$[\tilde{a}_{n+\ell/r}^a, \tilde{a}_{-m-\ell'/r}^b] = \left(n + \frac{\ell}{r}\right) C_{ab}\delta_{n,m}\delta_{\ell,\ell'}. \quad (4.18)$$

The correlation functions are

$$\langle \phi^a(w) \phi^b(w') \rangle = C_{ab} \log(w - w'), \quad (4.19)$$

$$\langle \varphi_\ell^a(w) \varphi_{\ell'}^b(w') \rangle = \delta_{\ell+\ell',r} C_{ab} \sum_{k=0}^{r-1} \omega^{-k\ell} \log \left[ 1 - \omega^k \left( \frac{w'}{w} \right)^{\frac{1}{r}} \right], \quad (4.20)$$

$$\langle \varphi^a(w) \varphi^b(w') \rangle = C_{ab} \log \left[ \frac{(1 - (w'/w)^{1/r})^r}{1 - (w'/w)} \right]. \quad (4.21)$$

For each  $e_a$ , we define

$$\psi_{e_a}(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{\ell=0}^{r-1} \omega^\ell : \exp \left[ \sqrt{\frac{1}{r}} \phi_a^{(\ell)}(w) \right] :, \quad (4.22)$$

where  $A_r$  is a normalization factor and

$$\phi_a^{(\ell)}(w) \equiv \varphi_a(e^{2\pi i \ell} w). \quad (4.23)$$

Let  $\alpha = \sum_{a=1}^{n-1} n_a e_a \in Q$ , where  $n_a$  are non-negative integers and  $Q$  denotes the root lattice. We obtain the corresponding parafermion, up to its normalization,

$$\psi_\alpha \sim \prod \psi_{e_a}^{n_a}. \quad (4.24)$$

The independent parafermion can be given only for the case  $\alpha \in Q/rQ$ . Not of all  $\psi_\alpha$  are independent;

$$1 \sim \underbrace{\psi_{e_a} \cdots \psi_{e_a}}_r. \quad (4.25)$$

For example, in the the case of  $\mathfrak{sl}(3)$  algebra and  $r = 4$ , the corresponding parafermions are drawn in the Fig. 1. We define the parafermion associated with negative of a simple root by

$$\psi_{-e_a} \sim \underbrace{\psi_{e_a} \psi_{e_a} \cdots \psi_{e_a}}_{r-1}. \quad (4.26)$$

The normalization can be determined by the correlation functions [47],

$$\langle \psi_\alpha(w) \psi_{-\alpha}(w') \rangle = (w - w')^{-2 + \frac{\alpha^2}{r}}, \quad (4.27)$$

where  $\alpha^2 = (\alpha, \alpha)$ . In particular,

$$\langle \psi_{e_a}(w) \psi_{-e_a}(w') \rangle = (w - w')^{-2 \frac{r-1}{r}}. \quad (4.28)$$



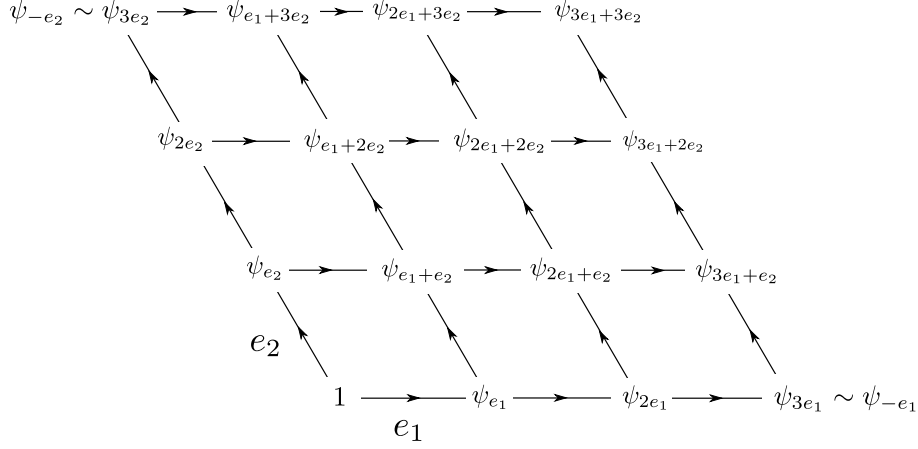


Fig. 1: The parafermions in the case of  $\mathfrak{sl}(3)$  and  $r = 4$ .

In the case of the  $\mathfrak{sl}(2)$  algebra, we obtain the first  $\mathbf{Z}_r$ -parafermion,

$$\psi_1(w) = \psi_{e_1}(w). \quad (4.29)$$

Similar to the case of  $n = 2$  (3.22), the central charge is given by

$$\begin{aligned} c_n^{(r)} &= \frac{n(n-1)(r-1)}{r+n} + (n-1) \left( 1 - n(n+1) \frac{Q_E^2}{r} \right) \\ &= \frac{r(n^2-1)}{r+n} - n(n^2-1) \frac{Q_E^2}{r}. \end{aligned} \quad (4.30)$$

When we set  $m = rm_+ + k$ ,  $m_- = m_+ + s$  in (4.8), this central charge becomes

$$\begin{aligned} c_n^{(r,m,s)} &= \frac{r(n^2-1)}{r+n} - \frac{rs^2n(n^2-1)}{m(m+rs)} \\ &= \frac{(n^2-1)r(\frac{m}{s}-n)(\frac{m}{s}+n+r)}{(r+n)\frac{m}{s}(\frac{m}{s}+r)}, \end{aligned} \quad (4.31)$$

which is the same as that of the coset model,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_{\frac{m}{s}-n}}{\widehat{\mathfrak{sl}}(n)_{\frac{m}{s}-n+r}}. \quad (4.32)$$

Compared with (1.1) we find

$$p = \frac{m}{s} - n. \quad (4.33)$$

In the case of  $s = 0$  corresponding to  $Q_E = 0$ , we have the central charge of the usual Sugawara stress tensor for  $\widehat{\mathfrak{sl}}(n)_r$ ,

$$c_n^{(r,m,0)} = \frac{r(n^2-1)}{r+n} = c_{\widehat{\mathfrak{sl}}(n)_r} \quad (4.34)$$

It is well-known that the affine Lie algebra  $\widehat{\mathfrak{sl}}(n)_r$  is represented by parafermions and an auxiliary boson [47]. In the case of  $s = 1$ , because (4.31) becomes

$$c_n^{(r,m,1)} = \frac{(n^2 - 1)r(m - n)(m + n + r)}{(r + n)m(m + r)}, \quad (4.35)$$

the model gives us the unitary series of the coset,

$$\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(n)_{m-n}}{\widehat{\mathfrak{sl}}(n)_{m-n+r}}. \quad (4.36)$$

We can see how the level  $p$  is related with the omega-background parameters  $\epsilon_1$  and  $\epsilon_2$  in the 4-d side. Since  $\beta = -\epsilon_1/\epsilon_2$ , (4.8) yields the condition to the ratio of these parameters. Therefore, when we introduce the free parameter  $\epsilon$ ,  $\epsilon_{1,2}$  can be written respectively as

$$\epsilon_1 = \epsilon(p + n + r), \quad \epsilon_2 = -\epsilon(p + n). \quad (4.37)$$

This result suggests that the Nekrasov-Shatashvili limit  $\epsilon_1 \rightarrow 0$  (resp.  $\epsilon_2 \rightarrow 0$ ) of the  $\mathcal{N} = 2$  gauge theory on the  $\mathbf{R}^4/\mathbf{Z}_r$  corresponds to the critical level limit  $p + r \rightarrow -n$  (resp.  $p \rightarrow -n$ ) of the coset model.

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